Fisher's scaling relation above the upper critical dimension

Kenna, R. and Berche, B.

Author post-print (accepted) deposited in CURVE January 2016

Original citation & hyperlink:
http://dx.doi.org/10.1209/0295-5075/105/26005

Publisher: IOP Publishing Ltd

ISSN 1434-6028
ESSN 1434-6036
DOI 10.1209/0295-5075/105/26005

Copyright © and Moral Rights are retained by the author(s) and/ or other copyright owners. A copy can be downloaded for personal non-commercial research or study, without prior permission or charge. This item cannot be reproduced or quoted extensively from without first obtaining permission in writing from the copyright holder(s). The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the copyright holders.

This document is the author's post-print version, incorporating any revisions agreed during the peer-review process. Some differences between the published version and this version may remain and you are advised to consult the published version if you wish to cite from it.

CURVE is the Institutional Repository for Coventry University
http://curve.coventry.ac.uk/open
Fisher’s scaling relation above the upper critical dimension

R. Kenna\textsuperscript{1} and B. Berche\textsuperscript{2}

\textsuperscript{1}Applied Mathematics Research Centre, Coventry University, Coventry, CV1 5FB, England
\textsuperscript{2}Statistical Physics Group, Institut Jean Lamour, UMR CNRS 7198, Université de Lorraine, B.P. 70239, 54506 Vandœuvre-lès-Nancy Cedex, France

PACS 64.60.-\textit{a} – General studies of phase transitions
PACS 05.20.\textit{\textminus}y – Classical statistical mechanics

Abstract – Fisher’s fluctuation-response relation is one of four famous scaling formulae and is consistent with a vanishing correlation-function anomalous dimension above the upper critical dimension \(d_c\). However, it has long been known that numerical simulations deliver a negative value for the anomalous dimension there. Here, the apparent discrepancy is attributed to a distinction between the system-length and correlation- or characteristic-length scales. On the latter scale, the anomalous dimension indeed vanishes above \(d_c\) and Fisher’s relation holds in its standard form. However, on the scale of the system length, the anomalous dimension is negative and Fisher’s relation requires modification. Similar investigations at the upper critical dimension, where dangerous irrelevant variables become marginal, lead to an analogous pair of Fisher relations for logarithmic-correction exponents. Implications of a similar distinction between length scales in percolation theory above \(d_c\) and for the Ginzburg criterion are briefly discussed.

Introduction. – The scaling hypothesis was developed half a century ago and stands as one of the pillars of modern theories of critical phenomena \cite{1}. In its basic form, six standard critical exponents are linked through four scaling relations \cite{2}. Of these, hyperscaling and Fisher’s fluctuation-response relation are notable in that the former involves the dimensionality \(d\) and the latter involves the anomalous dimension \(\eta\) \cite{2}. Above \(d = d_c\) dimensions, critical exponents assume their Landau values and hyperscaling was long considered to fail there (see e.g., \cite{3,4,5}). However, the introduction of a new, seventh, fundamental exponent \(ϙ\) (see footnote\textsuperscript{1}) extends hyperscaling beyond the upper-critical dimension \cite{2}. From Fisher’s dangerous-irrelevant-variables formalism, \(ϙ = d/d_c\) when \(d > d_c\). Below the upper critical dimension, \(ϙ\) reverts to 1. Evidence that \(ϙ\) is both physical and universal was given in Ref. \cite{2}. Physically, it is the exponent which governs the leading finite-size behaviour of the correlation length. Its universality is evidenced by finite-size scaling (FSS) at the pseudocritical point for both periodic and free boundary conditions \cite{3}.

The disentanglement of the correlation length from the actual length of the system when \(ϙ \neq 1\) is central to the extension of hyperscaling above \(d_c\) \cite{2}. Length scales also enter via the correlation function into the definition of the anomalous dimension and derivation of Fisher’s relation \cite{2}. Here we uncover associated subtleties above \(d_c\) which require re-interpretation of the formalism there. In particular, we show that the current paradigm, which does not distinguish length scales and their associate dimensionalities, violates bounds on the anomalous dimension, violates Fisher’s scaling relation, and leads to disparities between field-theoretic results in the thermodynamic limit and numerical simulations in finite volume. Our new formalism, which resolves all of these anomalies, entails two anomalous dimensions and two Fisher relations – one for each length scale. Additionally, the theory delivers two logarithmic analogues to the anomalous dimension and two corresponding relations at the upper critical dimension.

\textsuperscript{1} The notation \(\alpha, \beta, \gamma, \delta, \eta\) and \(\nu\) for the six primary critical exponents was standardised by Michael E. Fisher in the 1960’s. In Ref. \cite{2}, we introduced the new exponent \(ϙ\) as characterising the leading, power-law FSS of the correlation length, in analogy to the exponent \(q\), introduced in Ref. \cite{27}, which characterises the logarithmic-correction term there. Here we follow a suggestion by Fisher to switch to the archaic Greek letter \(ϙ\) (“koppa” or “oppa” – the source of Latin ““q””) to synchronise more closely with his standard nomenclature. We are grateful for this suggestion.
value of the temperature $T$ (e.g., defined as the location of the susceptibility peak in vanishing external field) and approaches the critical value $T_c$ as $L \to \infty$. The leading scaling behavior for the specific heat, spontaneous magnetization, susceptibility and correlation length are

$$c_c(t) \sim t^{-\alpha}, \quad m_c(t) \sim t^\beta, \quad \chi_c(t) \sim t^{-\gamma}, \quad \xi_c(t) \sim t^{-\nu},$$

respectively. The correlation function is usually written as $G(t, r) \sim r^{-\nu} \epsilon \frac{\epsilon}{\xi_c(t)}$, for a function $D$ which, for $r \gg \xi_c(t)$, decays exponentially, $D(y) \sim \exp(-y)$. When $r \ll \xi_c(t)$ the correlation function reduces to

$$G(t, r) \sim r^{-(d-2+\eta)},$$

to leading order. Above $d_c$, mean-field (MF) exponents describe scaling behaviour, and for the Ising model and associated $\phi^4$ theory, for which $d_c = 4$, these exponents are $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$, $\eta = 0$, $\nu = 1/2$. The standard hyperscaling relation,

$$\nu d = 2 - \alpha,$$

fails for $d > d_c$.

Eq. (1) may be expressed in terms of correlation length, e.g., $\chi_c(t) \sim \xi_c(t)^{\gamma/\nu}$. Since standard FSS is controlled by the ratio of the correlation length to the actual length, the replacement $\xi_c(t) \to \xi(t) \sim L$ then delivers the standard FSS formulae

$$c_L(0) \sim L^{\alpha/\nu}, \quad m_L(0) \sim L^{\beta/\nu}, \quad \chi_L(0) \sim L^{\gamma/\nu}. \quad (4)$$

Recently a seventh exponent was introduced which characterizes the FSS of the correlation length above, as well as below, $d_c$. [8] (see also Ref. [10]),

$$\xi_L(t) \sim L^\delta \quad \text{where} \quad \delta = \begin{cases} 1, & \text{if } d \leq d_c, \\ d/d_c, & \text{if } d \geq d_c. \end{cases} \quad (5)$$

This seventh exponent originates in Fisher’s dangerous-relevant-variable mechanism [10] provided an earlier assumption [11] that the finite-size correlation length $\xi_L$ is bounded by the length $L$ is relaxed. In Ref. [3], $\xi_L$ was referred to as a characteristic length. That is, it is fact, the finite-size correlation length was established directly in Refs. [5,12] for periodic boundary conditions and indirectly in Ref. [5] for free boundaries. The exponent $\delta$ extends hyperscaling and FSS beyond the upper critical dimension via the relation

$$\nu d/\delta = 2 - \alpha,$$

and is supported analytically [5,13] and numerically [5,12].

Instead of being governed by the ratio of two length scales $\xi_c/L$, FSS now emerges through the ratio of the correlation volume in $d_c$ dimensions to actual volume, namely $\xi^{d_c}_c/L^d$. In other words the usual prescription is replaced by $\xi_c \to \xi_L = L^\delta$, which, from Eq. (4) delivers

$$c_L(0) \sim L^{\alpha/\nu}, \quad m_L(0) \sim L^{-\beta/\nu}, \quad \chi_L(0) \sim L^{\gamma/\nu}. \quad (7)$$

Eq. (7), termed $Q$-FSS in Ref. [3] to compactly distinguish it from Eq. (4), has been verified for systems with periodic and free boundary conditions [5,12,14].

Here we turn our attention to Fisher’s fluctuation-response relation. The standard derivation starts from the fluctuation-dissipation theorem, viz.

$$\chi_L(t) \sim \int_a^L G(t, r) r^{d-1} dr. \quad (8)$$

Here $a$ is the lattice constant in condensed matter, vanishing in the continuum field theory. Close to criticality, where $t$ is sufficiently small, so that $r \ll \xi$, $G(t, r) \sim r^{-(d-2+\eta)} D(r/\xi(t))$, and in the thermodynamic limit, Eq. (8) becomes

$$\chi_L(t) \sim \int_a^\infty G(t, r) r^{d-1} dr. \quad (10)$$

We partition this as

$$\chi_L(t) \sim \int_a^{S\xi(t)} D(r/\xi(t)) \frac{dr}{r^{d-1}} + \int_{S\xi(t)}^\infty G(t, r) r^{d-1} dr, \quad (11)$$

where $S$ is a constant. The second term is assumed to give rise to additive corrections close to criticality, where $\xi(t)$ diverges. The first term gives

$$\chi_L(t) \sim \int_a^{S\xi(t)} D(y) y^{1-\eta} dy. \quad (12)$$

One assumes that the lower integral limit only contributes to additive corrections to scaling, yielding to leading order,

$$\chi_L(t) \sim \xi^{-\eta}_L(t). \quad (13)$$

Eq. (11) then gives Fisher’s relation [3],

$$\eta = 2 - \gamma/\nu. \quad (14)$$

If $L$ is finite, on the other hand, a similar procedure gives (setting $a = 0$ to extract the leading scaling)

$$\chi_L(t) \sim \xi^{-\eta}_L(t) \int_0^S D(y) y^{1-\eta} dy, \quad (15)$$

where $S = L/\xi_L(t)$. Provided $\xi_L(0) \sim L^\delta$, the standard FSS formulae [3] deliver $\chi_L(0) \sim \xi^{-\eta}_L(0)$, which again recovers Fisher’s relation.

**Inconsistencies Above the Upper Critical Dimension.** – The above derivation of Fisher’s scaling relation for finite-size systems runs into trouble if $d > d_c$. There, with $\xi_L(0) \sim L^\delta$, the upper integral limit in Eq. (15) has a leading $L$-dependency, destroying the derivation even with the $Q$-FSS form for $\chi_L$.

To investigate further, we simulated the $d = 5$ Ising model for periodic lattices [8]. Denoting the Ising spin
where \( u \) is the scaling field associated with the coefficient of the quartic term in the Landau expansion. Above \( d_c \), the critical behaviour is controlled by the Gaussian fixed point in the renormalization-group formalism, where

\[
y_t = 2, \quad y_h = 1 + d/2, \quad y_u = 4 - d. \tag{17}
\]

Because of a discrepancy between \( \alpha, \beta \) and \( \delta \) coming from directly differentiating \( \eta \) and MF estimates, Fisher introduced the notion of dangerous irrelevant variables for the free energy density in the thermodynamic limit \([11]\).

When \( u \to 0 \), Eq. \((10)\) becomes \([11]\)

\[
f_L(t, h, u) = b^{-d} f_{L/f}(tb^{y_h}, hb^{y_u}, ub^{y_u}) = L^{−d/2} f_1 \left( tL^{y_h}, hL^{y_u} \right), \tag{18}
\]

where \( y_t = y_t + p_2y_u \) and \( y_u = y_h + p_3y_u \). Landau exponents are recovered if \( p_2 = -1/2 \) and \( p_3 = -1/4 \) \([11] [20]\).

No similar dangerous-irrelevant-variable mechanism was expected for the correlation length or correlation function, since MF theory and Gaussian fixed-point values of \( \gamma, \eta \) and \( \nu \), which are all connected to the correlation function, agree. Notwithstanding this, similar considerations for the correlation length deliver

\[
\xi_L(t, h, u) = L^{y_x} \left( tL^{y_h}, hL^{y_u} \right). \tag{19}
\]

In Ref. \([11]\), \( y \) was set to 1 in Eq. \((10)\) because of an expectation that \( \xi_L \) is bounded by \( L \). For this reason, another length scale, \( \xi_{\text{full}} \sim t^{-1} y^{*}_t \), was introduced in such a way that the first argument on the right-hand side of Eq. \((19)\) involves a ratio \( \xi_{\text{full}}(t)/L \) which governs FSS. See also Ref. \([11] [21]\).

Luijtjen and Blöte obtained the FSS of the correlation function by differentiating Eq. \((15)\) with respect to two local magnetic fields \( h(0) \) and \( h(r) \) \([15]\). When dangerous irrelevant variables are (incorrectly) not accounted for, \( y_t^* \) and \( y_u^* \) are replaced by \( y_t \) and \( y_u \), respectively, so that \( G \propto L^{2(y_t^* - d)} = L^{-(d-2)} \), which is the standard, Landau, MF result with \( \eta = 0 \). However, (correctly) taking account of the dangerous irrelevancy in Eq. \((18)\), Luijtjen and Blöte obtained instead \( G \propto L^{2(y_u^* - d)} = L^{-(d-2)} \), corresponding to an anomalous dimension \( \eta^* \equiv 2 - d/2 \).

Luijtjen and Blöte give a second interpretation to their anomalous dimensions \([15]\). Writing the Ginzburg-Landau-Wilson \( \phi^4 \) type \([3][10][17]\). How the Standard Paradigm Addresses the Problem of the Negative Anomalous Dimension.

The problem of the negative anomalous dimension was already noticed by Nagle and Bonner over 40 years ago \([15]\). In a numerical study, they determined the correlation decay in a spin chain with long-range interactions and measured an anomalous dimension different from the standard one. In an attempt to explain this, Baker and Golner analytically determined spin-spin correlations in an Ising model for which scaling is exact \([19]\). Their explanation was that “long long-range order” is controlled by a different anomalous dimension to the standard one, which controls “short long-range order”. They found that the long long-range exponent fails to satisfy the scaling relation for the anomalous dimension above the upper critical dimension.

The problem was revisited over a decade ago in a series of papers by Luijtjen and Blöte \([15]\). To recount their analysis, we follow Fisher and first write the scaling form for the free energy density as \([10]\)

\[
f_L(t, h, u) = b^{-d} f_{L/f}(tb^{y_h}, hb^{y_u}, ub^{y_u}), \tag{16}
\]

Figure 1: The power-law decay of the correlation function for the \( d = 5 \), critical and pseudocritical Ising model favors \( G_Q(t, L/2) \sim L^{-5/2} \). The insert shows that the effective exponent approaches the \( Q \)-theoretic value \( \eta_Q = -1/2 \) as the minimum lattice size used in the fit \( L_{\text{min}} \) increases.

at site \( i \) of the lattice by \( S_i \), the correlation function is \( G(t, i) = \langle S_0(S_0) - \langle S_0 \rangle^2 \rangle \). To extract the exponent \( p \) from the general form \( G(t, r) \sim D(r/L)^{-p} \) at criticality, \( G(t, L/2) \) is plotted against \( L \) in Fig. 1 \([15]\). The result clearly supports \( p = 5/2 \). If, as the standard paradigm purports, \( p = d - 2 + \eta \), this would correspond to a value of the anomalous dimension of \(-1/2 \).

A negative value for the anomalous dimension poses problems. Firstly it is in disagreement with mean-field theory and Landau theory, which deliver \( \eta = 0 \). Secondly, it violates Fisher’s scaling relation \([14]\).

Thirdly, it appears to violate field theory which delivers a non-negative \( \eta \) whereas the \( \eta \)-theoretic value \( \eta_Q \) is recovered if \( \eta = -1 \). If, as the standard paradigm \([3][16][17]\), one sets \( \eta = 0 \) in Eq. \((19)\) because of an expectation that \( \xi_L \) is bounded by \( L \), this would correspond to a value of the anomalous dimension of \(-1/2 \).

In Ref. \([11]\), \( \eta \) was set to 1 in Eq. \((15)\) because of an expectation that \( \xi_L \) is bounded by \( L \). For this reason, another length scale, \( \xi_{\text{full}} \sim t^{-1} y^{*}_t \), was introduced in such a way that the first argument on the right-hand side of Eq. \((19)\) involves a ratio \( \xi_{\text{full}}(t)/L \) which governs FSS. See also Ref. \([11] [21]\).

Luijtjen and Blöte obtained the FSS of the correlation function by differentiating Eq. \((15)\) with respect to two local magnetic fields \( h(0) \) and \( h(r) \) \([15]\). When dangerous irrelevant variables are (incorrectly) not accounted for, \( y_t^* \) and \( y_u^* \) are replaced by \( y_t \) and \( y_u \), respectively, so that \( G \propto L^{2(y_t^* - d)} = L^{-(d-2)} \), which is the standard, Landau, MF result with \( \eta = 0 \). However, (correctly) taking account of the dangerous irrelevancy in Eq. \((18)\), Luijtjen and Blöte obtained instead \( G \propto L^{2(y_u^* - d)} = L^{-(d-2)} \), corresponding to an anomalous dimension \( \eta^* \equiv 2 - d/2 \).

Luijtjen and Blöte give a second interpretation to their anomalous dimensions \([15]\). Writing the Ginzburg-Landau-Wilson action in momentum space,

\[
F[\phi] = \frac{1}{2} \sum_k (k^2 + \xi^{-2}) |\phi_k|^2 + \frac{u}{4L^d} \sum_{k_1, k_2, k_3} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \tag{20}
\]

where \( k_4 = -k_1 - k_2 - k_3 \). Ignoring the danger by setting \( u \) to zero, the correlation function is identified as the inverse of the quadratic part of the action, leading again to the Ornstein-Zernike expression,

\[
G^{-1}(t, k) = k^2 + \xi^{-2}(t). \tag{21}
\]

From the general form \( G^{-1}(t, k) = k^2 + \xi^{-2}(t) \), one identifies the Gaussian value \( \eta = 0 \). The same result is
obtained from Eq. (20) by first taking the thermodynamic limit \( L \to \infty \).

Keeping the quartic term in Eq. (20) with finite \( L \), however, the full quadratic part is

\[
\frac{1}{2} \sum_k \left( k^2 + \xi^{-2} + \frac{3u}{2L^d} \phi_0^2 \right) |\phi_k|^2,
\]

where \( \phi_0 \) is the zero mode associated with periodic boundary conditions. Since \( \langle \phi_0^2 \rangle \) behaves as \( \chi_L \sim L^{d/2} \), the final term in parentheses is \( L^{-d} \phi_0^2 \sim L^{-d/2} \), which we identify as \( k_{\min}^{d/2} \) acting as an additional momentum term. It was argued in Refs. [15] that this term dominates large distance behaviour in Eq. (22), leading to \( \eta^* = 2 - d/2 \).

To summarise, in the standard paradigm there is a discrepancy between the Landau MF value \( \eta = 0 \) for the anomalous dimension above \( d_c \), and the value \( \eta^* = 2 - d/2 \), measured on finite systems. On the one hand this discrepancy is linked to neglecting, or accounting for, the dangerous irrelevant variable \( u \) (leading to \( \eta = 0 \) or \( \eta^* = 2 - d/2 \), respectively). On the other hand it is attributed to a difference between short-long range (\( \eta = 0 \)) and long-long range behaviour (\( \eta^* = 2 - d/2 \)).

The standard paradigm does not, however, explain how \( \eta^* = 2 - d/2 \) for long distance is manifest as \( \eta = 0 \) in the infinite-volume limit where field-theoretic theorems outlawing negative anomalous dimensions apply. Nor does it explain why it is the correct, dangerous-irrelevant-variables, long-long range \( \eta^* \) which conflicts with Landau and MF theory, fails to satisfy Fisher’s relation and violates field theory. (One would rather expect the conflict to be associated with the incorrect processes of neglecting dangerous irrelevant variables or taking short rather than long long distances.) Therefore the standard paradigm does not explain scaling above the upper critical dimension.

Resolution of Puzzle. – Here we offer an alternative explanation for the negativity of the measured value of the anomalous dimension, based on the \( Q \)-theory proposed in Ref. [3, 4]. This new explanation also resolves all of the above puzzles. According to the theory, there is a difference between the underlying length scale \( L \) of the system above \( d_c \) and its correlation length scale \( \xi_L \). This difference is manifest as \( \xi_L \sim L^{\eta} \).

In Eq. (9), the distance \( r \) is implicitly measured on the correlation length scale and this leads to the usual Fisher relation. In Eq. (15), however, the length scales \( L \) and \( \xi_L \) are incorrectly mixed above the upper critical dimension.

To repair this, we write the correct correlation function in terms of the system-length scale as

\[
G_Q(0,r) \sim r^{-(d-2+\eta_Q)} D_Q(r/L),
\]

where \( \eta_Q \) is the anomalous dimension measured on this scale, the subscript indicating that \( Q \)-FSS rather than standard FSS prevails there [8].

Eq. (15) for the susceptibility is then

\[
\chi_L(0) \sim \int_0^L r^{1-\eta_Q} D_Q \left( \frac{r}{L} \right) dr = L^{2-\eta_Q} \int_0^L D_Q(y) y^{1-\eta_Q} dy.
\]

Above \( d = d_c \), the \( Q \)-FSS formulae then yield

\[
\eta_Q = 2 - \gamma \nu/\nu.
\]

In the Ising case, where \( \gamma = d/4, \nu = 1 \) and \( \nu = 1/2 \), this gives \( \eta_Q = 2 - d/2 \) and identifies \( \eta_Q \) with \( \eta^* \) of Refs. [15, 18, 19]. Eq. (20) is the fluctuation-response relation above the upper critical dimension when distance is measured on the scale of system size. The standard expression (14) is the equivalent formula there when distance is measured on the correlation-length scale. The relationship between the two anomalous dimensions is then

\[
\eta_Q = \gamma \nu + 2(1 - \gamma).
\]

Below \( d_c \), the two anomalous dimensions coincide. When \( d > d_c \), \( \eta_Q \) is negative. Since the non-negativity bounds for the anomalous dimension refer to correlation decay on the scale \( L \), they involve \( \eta \) rather than \( \eta_Q \), and are not violated [3, 5, 6, 16, 17, 22].

This interpretation advocates that there are two forms for the correlation function, two anomalous dimensions and two Fisher relations, depending on whether distance is measured on the scale of \( L \) or \( \xi_L \sim L^\eta \). The value \( \eta = 0 \) is correct when distance is measured on the scale of \( \xi_L \) and \( \eta_Q = 2 - d/2 \) is correct on the length-scale \( L \). Both are valid as characterising long-distance decay. On either scale, there is no need to distinguish between short long distances and long long distances. Our numerical results for the short-range model, and Luijten’s and Blöte’s numerics for its long-range counterpart, confirm \( \eta_Q \) or \( \eta^* = 2 - d/2 \) as governing the correlation decay at criticality [14].

Logarithmic Corrections at \( d_c \). – Thus, there is no numerical disagreement between our results and those of Refs. [15, 18, 19] for \( d > d_c \). At this point neither theory is falsified by numerics. Instead, interpretations differ. But these interpretations are important at a fundamental level. While each interpretation can be couched in terms of the dangerous-irrelevant-variables mechanism, the paradigm hitherto relies completely on the role of the quartic term. To discriminate between them, we need a scenario without dangerous irrelevant variables and \( d = d_c \) presents such a case. There \( u \) is marginal with logarithmic corrections arising from the renormalization-group formalism. We shall now show that, while the Luijten-Blöte scenario has no consequence at \( d = d_c \), our scaling theory again leads to two correlation functions and to logarithmic analogues to each of the Fisher relations (13) and (25). This provides a route to test interpretations numerically.

We follow the notation of Ref. [23] and denote the logarithmic-correction exponents, that are known to ap-
pear at $d = d_c$, by hatted indices,
\[ \chi_\infty(t) \sim t^{-\gamma} \ln t^{\delta}, \]
\[ \xi_\infty(t) \sim t^{-\nu} \ln t^{\delta}, \]
\[ \xi_L(0) \sim L (\ln L)^{\hat{\nu}}. \]

For the Ising and $\phi^4$ models, $\gamma = 1/3$, $\nu = 1/6$, $\hat{\nu} = 1/4$ [13][23]. (The exponent $\hat{\nu}$ was written $\tilde{\nu}$ in Ref. [24].) The correlation function in the critical region is
\[ G(0, r) \sim D[y(r)] r^{-(d-2+\eta)} (\ln r)^{\eta}, \]
where $y(r) = r/\xi_L(0)$ if distance is measured on the correlation-length scale. If the system-length scale is used instead, then $y(r) = r/L$ and $\hat{\eta}_Q$ replaces $\hat{\nu}$ in Eq. (30).

For the $d = 4$ Ising model $\eta = 0$ [24]. With logarithmic corrections, Eq. (13) and Eq. (21) become
\[ \chi_\infty(t) \sim \xi_\infty^{-\eta} (t) [\ln \xi_\infty(t)]^{\eta} [1 + \mathcal{O}(1/\ln \xi_\infty)], \]
\[ \chi_L(0) \sim L^{2-\eta_Q} \ln(L)^{\eta_Q} [1 + \mathcal{O}(1/\ln L)]. \]

Inserting Eqs. (27) and (28) and their FSS counterparts, respectively, yields the analogues to Eqs. (14) and (25),
\[ \gamma = (2-\eta) \hat{\nu} + \hat{\eta}, \]
\[ \hat{\gamma} = (2-\eta_Q) (\nu - \hat{\nu}) + \hat{\eta}_Q. \]

Of course, $\eta_Q = \eta$ in Eqs. (30)-(34) since $d = d_c$ there. The relation (33) is the same as that proposed in Ref. 23.

Indeed, in Ref. 24, this formula was verified in a variety of models at their respective upper critical dimensions in the infinite-volume limit, through exponential decay of the correlation function, i.e., where distance is measured in units of the correlation length.

However, finite-size numerical approaches are defined on the underlying lattice with length-scale $L$, for which
\[ \hat{\eta}_Q = \hat{\eta} + (2-\eta) \hat{\nu}. \]

Thus $\hat{\eta}_Q = 1/2$ in the $d = 4$ Ising model. We test these predictions in Fig. 2 where $(L/2)^2 G(t, L/2)$ is plotted against $\ln (L/2)$ at both the critical and pseudocritical points. The positive slope is clearly not $\hat{\eta} = 0$. Compatibility with $\hat{\eta}_Q = 1/2$ is evident and fits to $A (\ln (L/2 + B))^{1/2}$, both at criticality and at pseudocriticality, are nicely compatible with the numerical data.

**Discussion.** Returning to $d > d_c$ case, our claim is that both $\eta$ and $\eta_Q$ are valid at long distances. For this claim not to violate Fisher’s dangerous-irrelevant-variables theory, $\eta$ should also arise from, just as $\eta_Q$ does. Indeed we can see the emergence of both anomalous dimensions through the scaling of the correlation function in a manner similar to the development of Eqs. (15) and (19) above.

From dimensional analysis, one may write the standard form
\[ G_L(t, u, r) = b^{2X} G_{L/4} \left( (b^{\nu} \ast u b^{\nu} \ast rb^{-1}) \right), \]

in which $X_0 = d/2 - 1$. Note that $r$ and $b$ have the same dimension as $L$. Acknowledging the danger of $u$, we treat this in a similar manner to Eqs. (15) and (19) and write
\[ G_L(t, u, r) = b^{2-\delta v} G_{L/2} \left( (b^{\nu} \ast rb^{-1+y \ast v}) \right). \]

Interpreting $r$ as a length requires $v_2 = 0$ to render the final argument on the right dimensionless. With $t = 0$ and $b = r$, we then obtain $G_L(0, u, r) = 1/r^{d-2-\delta v}$, which accords with $G_Q$ in Eq. (28) provided that $v_1 = -\eta_Q/y = -1/2$. If, on the other hand, we interpret $r$ as a correlation length, the final argument is dimensionless if it is $rb^{-\hat{\nu}}$. We then require $v_2 = (1 - q)/y = 1/4$. Again setting $t = 0$, but now setting $b^\delta = r$, we obtain the scaling of the correlation function as $G_L(0, u, r) = 1/r^{(d-2-\delta v_1)/\hat{\nu}}$. Inserting $v_1 = -1/2$ delivers the Ornstein-Zernike form
\[ G(0, u, r) \sim 1/r^2. \]

In conclusion, for a comprehensive picture of scaling above the upper critical dimension, one must take care whether distance is measured in terms of the system-length scale or the correlation-length scale. To track these, two correlation functions are required, resulting in two Fisher relations, involving two anomalous dimensions, *only one of which is captured by Landau theory and MF theory*. The hidden anomalous dimension is revealed through numerical simulations on the system-length scale. At the upper critical dimension itself, analogous expressions arise for the logarithmic corrections to scaling there.

The magnetisation transitions in spin models are equivalent to percolation transitions of Fortuin-Kasteleyn clusters. The 30-year-old prevailing picture of hyperscaling breakdown in percolation theory predicts that the number of spanning clusters $N_T$ is finite for $d < d_c$ but diverges as $N_T \sim L^{d-d_c}$ for $d > d_c$ ($d_c = 6$ for percolation theory) [24]. The theory also predicts that the critical clusters have fractal dimension $D = (\beta + \gamma)/\nu$, which is independent of $d$ when $d > d_c$. This perceived clear demarcation...
between $d < d_c$ and $d > d_c$ has been steadily undermined over the years [23,24]. In Ref. [24], because finite-size simulations did not follow the standard theory, and $N_L \sim L^d$ is claimed instead depending on boundary conditions, the behaviour of $N_L$ above $d_c$ was declared an “open issue”.

A simple thought experiment shows that the standard interpretation of $N_L \sim L^d$—spanning clusters is flawed. With interactions of sufficiently long range, one can construct a percolation or spin model with $d_c < 1$. That result then predicts a diverging number of spanning clusters in $d = 1$ dimension despite there being only enough physical space to accommodate one such cluster there. There can, however, be a (finite) number of critical clusters of length $O(L)$.

The fundamental error undermining prevailing percolation theory above $d_c$ is the assumption that $\xi_L \sim L$ ($\xi$ is the connectedness length in pure percolation theory). Using Eq. (6) instead, carefully distinguishing finite $L$ from its infinite limit, and otherwise following Ref. [24], one derives Eq. (6) for all $d$. This approach delivers $N_L \sim L^d$, compatible with the above thought experiment and with the aforementioned claim in Ref. [25]. Q-theory also predicts that the mass of the critical clusters is $\xi_q^d = L^{D_Q}$ where $D_Q = dD$. The fractal dimension of the critical clusters is therefore independent of $d$ only when measured on the correlation-length scale above $d_c$.

It is also legitimate to ask, in the present framework, about the status of the Ginzburg criterion, which defines $d_c$ as that dimension above which fluctuations become negligible and Landau exponents prevail. It is usually obtained, for example, by comparing the fluctuations, measured by $\chi$, with the average magnetization-squared both at the correlation-length scale. The standard argument is that, for MF theory to be correct, one should have $\chi \ll m^2\xi^d$ or $d > d_c = (\gamma + 2\beta)/\nu$. This defines $d_c = 4$ when Landau exponents are used. Since we now know that above $d_c$ the correlation length exceeds the system size, the above argument is valid only at the scale $L$, where fluctuations $\chi_L \sim L^{\gamma/\nu}$ now appear to be of the same order as the average square $m^2\xi^d \sim L^{d-2\beta/\nu}$ when Landau exponents are plugged in. This shows that the correlations have not been washed out at the size $L$ (but the correlators still decay as $r^{-d/2}$).

Strictly speaking, MF theory is not fully valid above $d_c$: while the thermal exponents $\alpha$, $\beta$, $\gamma$ and $\nu$, and the magnetic counterpart $\delta$ are those of Landau theory, the exponent describing the space dependence of the correlation function is $\eta_q$ rather than $\eta$, which describes an emergent 4-dimensional field theory at the scale of the correlation length.

* * *

We thank M.E. Fisher, Yu. Holovatch, F. Iglói and N. Izmailian for careful readings of the manuscript and helpful discussions. We also thank M.E. Fisher for suggesting to introduce the symbol $\Psi$ for the new exponent and J. Cardy for advice on its typesetting. We also thank F. Iglói for suggestion to include material on percolation theory and J.-C. Walter for help with the numerics This research was supported by Marie Curie IIF and IRSES grants within the 7th EU Framework Programme.

References
